Generating Functions

A Gentle Introduction to a Different Way of Counting

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A generating function is a clothesline on which we hang up a sequence of numbers for display.

— Herbert S. Wilf

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Prerequisites

- I will try to make this talk accessible as possible
- A highschool level of combinatorics and a calculus 2 level of sequences and series are assumed
- But the most important thing is understanding, so feel free to ask questions during the talk

What are generating functions?

- A way to encode a sequence of numbers into a function
- First used in the 1730s by Abraham de Moivre to solve a linear recurrence
- Made rigorous by studying the “ring of formal power series”
- But we will ignore most formalities and just get to the interesting applications

Ordinary Generating Functions

Definition

Given a sequence $\{a_n\}_{n=0}^{\infty}$, its ordinary generating function is the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Example

Sequence $1, 1, 1, 1, \ldots$ has ogf $f(x) = 1 + x^1 + x^2 + x^3 + \cdots = \frac{1}{1-x}$

Example

Sequence $0, 1, 2, 3, \ldots$ has ogf $f(x) = x^1 + 2x^2 + 3x^3 + \cdots$

Notation

Relating a sequence with its function

Given a sequence $\{a_n\}_{n=0}^{\infty}$ with ordinary generating function $f(x)$, we can write

$$\{a_n\}_{n=0}^{\infty} \stackrel{\text{ogf}}{\longleftrightarrow} f(x)$$

A function's coefficient

If $f(x)$ is a function with power series representation as $\sum_{n=0}^{\infty} a_n x^n$, then we use the following symbol for its coefficient.

$$[x^n]f(x) = a_n$$

Multiplying Two Ordinary Generating Functions

Suppose $f(x) = a_0 + a_1x + a_2x^2 + \dots$ and $g(x) = b_0 + b_1x + b_2x^2 + \dots$. Then,

$$(fg)(x) = a_0b_0 + (a_0b_1 + a_1b_1)x + (a_0b_2 + a_1b_1 + a_2b_2)x^2 + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + \dots$$

Multiplying Two Ordinary Generating Functions

The Cauchy Product

If $f(x) \overset{\text{ogf}}{\longleftrightarrow} \{a_n\}_{n=0}^{\infty}$ and $g(x) \overset{\text{ogf}}{\longleftrightarrow} \{b_n\}_{n=0}^{\infty}$, then

$$(fg)(x) \overset{\text{ogf}}{\longleftrightarrow} \left\{ \sum_{k=0}^{n} a_k b_{n-k} \right\}_{n=0}^{\infty}.$$

Or alternatively,

$$[x^n]fg(x) = \sum_{k=0}^{n} a_k a_{n-k}.$$

A Fruit Counting Example

We want to make a fruit basket with 11 total fruit with:

- 2 or more oranges
- 3 or more odd count strawberries
- either 3 or 4 watermelons

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We can use generating functions coefficients to indicate whether or not we can have a certain number of fruits. For example, because we can only use even 2 or more oranges, our sequence is $0, 0, 1, 1, 1 \ldots$.

$$f(x) = x^2 + x^3 + x^4 + \ldots \quad g(x) = x^3 + x^5 + x^7 + \ldots \quad h(x) = x^3 + x^4$$

A Fruit Counting Example

We can multiply all the equations together to a generating function where $[x^n](fgh)(x)$ represents the number of fruit baskets with n fruit. When multiplying the functions, for any $d_k x^k$ in the final function, we are adding up x^k a total of d_k times because all the coefficients are 1. Hence, we are counting the number of ways to add up to k by the ways we can get $x^\alpha x^\beta x^\gamma = x^k$.

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$$f(x) = \frac{x^2}{1-x} \quad g(x) = \frac{x}{1-x^2} \quad h(x) = x^3 + x^4$$

$$\begin{aligned}(fgh)(x) &= \frac{x^2}{1-x} \cdot \frac{x}{1-x^2} \cdot (x^3 + x^4) \\ &= \frac{x^6}{(1-x)^2}\end{aligned}$$

A Fruit Counting Example

Since we are interested in a basket with 11 fruit, we want $[x^{11}](fgh)(x)$. How do we find this? Can use any method you want to find the Taylor series coefficient.

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Input interpretation

series	$\frac{x^6}{(1-x)^2}$	point	$x = 0$

Series expansion at x=0

$$x^6 + 2x^7 + 3x^8 + 4x^9 + 5x^{10} + 6x^{11} + O(x^{12})$$

(Taylor series)

(converges when $|x| < 1$)

$$[x^{11}](fgh)(x) = 6$$

An Integer Partitions Example

An integer partition of a natural number n is a way of writing n as a sum of positive integers.

Let P_n be the number of distinct integer partitions of n. Starting with $p_0 = 1$, the first few numbers of the sequence are

$$1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 10, \ldots$$

An Integer Partitions Example

Definition

$P_{n,\text{distinct}}$ is the number of integer partitions with distinct summands in its sum.

Definition

$P_{n,\text{odd}}$ is the number of integer partitions with an odd number of summands in its sum.

$P_{n,\text{distinct}} = P_{n,\text{odd}}$ for all n

We will show this by first calculating the generating functions for the sequences separately, then showing the generating functions are equal. When generating functions are equal, the sequences are equal.

An Integer Partitions Example

We examine how to construct the ogf for integers without restrictions, we later see that adding restrictions is easy from this starting point.

$$\{p_n\}_{n=0}^{\infty} \overset{\text{ogf}}{\longleftrightarrow} \sum_{n=0}^{\infty} p_n x^n$$
$$= (1 + x^1 + x^{1+1} + \dots)(1 + x^2 + x^{2+2} + \dots)(1 + x^3 + x^{3+3} + \dots)\dots$$
$$= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\dots$$

In the first factor, 1 counts partitions using zero 1's, x^1 counts using one 1's, x^{1+1} counts using two 1's, etc.
In the second factor, 1 counts partitions using zero 2's, x^1 counts using one 2's, x^{2+2} counts using two 2's, etc.
So on and so forth.

An Integer Partitions Example

$$
\begin{aligned}
\{p_{n,\text{distinct}}\}_{n=0}^{\infty} &\overset{\text{ogf}}{\longleftrightarrow} (1+x)(1+x^2)(1+x^3)\dots \\
&= (1+x)\frac{(1-x)}{(1-x)}(1+x^2)\frac{(1-x^2)}{(1-x^2)}(1+x^3)\frac{(1-x^3)}{(1-x^3)}\dots \\
&= \frac{(1-x^2)(1-x^4)(1-x^6)\dots}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\dots} \\
&= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^5}\right) \\
&\overset{\text{ogf}}{\longleftrightarrow} \{p_{n,\text{odd}}\}_{n=0}^{\infty}
\end{aligned}
$$

Exponential Generating Functions

Definition

Given a sequence $\{a_n\}_{n=0}^{\infty}$, its exponential generating function is the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

We also write $f(x) \overset{\text{egf}}{\longleftrightarrow} \{a_n\}_{n=0}^{\infty}$.

Why?

- sometimes ordinary generating functions do not converge to a nice analytical function
- multiplying or composing exponential generating functions can sometimes give nicer functions to work with

Exponential Generating Functions

Example

The number of n bit binary strings is given by sequence $2^0, 2^1, 2^2, 2^3, \ldots$. The exponential generating function $f(x)$ of the sequence is

$$\begin{aligned} f(x) &= 2^0\frac{x^0}{0!} + 2^0\frac{x^1}{1!} + 2^1\frac{x^2}{2!} + \ldots \\ &= \sum_{k=0}^{\infty} 2^n \frac{x^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(2x)^n}{n!} \\ &= e^{2x} \end{aligned}$$

So we get a nice closed form. (No nice closed form can be said for this sequence with ogf's.)

Exponential Generating Functions

Suppose $f(x) = a_0 + a_1 x + \frac{a_2}{2!}x^2 + \dots$ and $g(x) = b_0 + b_1 x + \frac{b_2}{2!}x^2 + \dots$. We get, $(fg)(x) = c_0 + c_1 x + c_2 x^2 + \dots$.

$$\begin{aligned} c_k &= \sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k!)} \\ &= \frac{1}{n!}\left(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a_k b_{n-k}\right) \\ &= \frac{1}{n!}\left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\right) \end{aligned}$$

Because we divide the sequence by $n!$ in egf's, we have the corresponding sequence being the value in the parentheses.

Exponential Generating Functions

Multiplicative Property

If $f(x) \overset{\text{egf}}{\longleftrightarrow} \{a_n\}_{n=0}^{\infty}$ and $g(x) \overset{\text{egf}}{\longleftrightarrow} \{b_n\}_{n=0}^{\infty}$, then

$$(fg)(x) \overset{\text{egf}}{\longleftrightarrow} \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right\}_{n=0}^{\infty}.$$

A String Derangement Example

Definition

Given a string of n characters, a *derangement* of the string is a permutation of the string where there are no fixed points.

i.e If we are given string S, a derangement of S is a permutation T of S such that $S[i] \neq T[i]$ for all i.

How many derangements D_n are there of a string of n unique letters?

A String Derangement Example

Let $\{D_n\}_{n=0}^{\infty} \overset{\text{egf}}{\longleftrightarrow} D(x)$. The number of permutations of n letters with exactly k fixed points is D_{n-k}. For a fixed k, there are $\binom{n}{k}$ ways to pick the fixed points. Hence, there are $\binom{n}{k} D_{n-k}$ permutations of n letters with k fixed points.

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We sum over all k to get a possible permutations. Another way to count all possible permutations of n letters is $n!$. Hence,

$$n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k}.$$

A String Derangement Example

$$n! = \sum_{k=0}^{n} \binom{n}{k} 1 \cdot D_{n-k}.$$

We multiply a 1 in to make application of the product rule more clear. Recalling the product rule, we see that RHS corresponds to the product of the egf's for the sequences $1, 1, 1, \ldots$ and $D_0, D_1, D_2, \ldots$. The two egf's are e^x and $D(x)$ respectively.

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Looking at the LHS, the sequence $0!, 1!, 2!, 3!, \ldots$ has egf $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$. Hence, we get the equality,

$$\frac{1}{1-x} = e^x D(x)$$

A String Derangement Example

Solving the equation, $D(x) = e^{-x}\frac{1}{1-x}$.

$$
\begin{aligned}
D(x) &= e^{-x}\frac{1}{1-x} \\
&= (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots)(1 + x + x^2 + x^3 + \dots) \\
&= 1 + (1 - 1 + \frac{1}{2!})x^2 + (1 - 1 + \frac{1}{2!} - \frac{1}{3!})x^3 + \dots
\end{aligned}
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&= 1 + (1 - 1 + \frac{1}{2!})x^2 + (1 - 1 + \frac{1}{2!} - \frac{1}{3!})x^3 + \dots
\end{aligned}
$$

In general, $[x^n]e^{-x}\frac{1}{1-x} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots (-1)^n\frac{1}{n!}$.
Hence, we get

$$D_n = n!\left(\sum_{k=0}^{n}(-1)^k\frac{1}{k!}\right).$$

Dirichlet Generating Functions

Definition

Given a sequence $\{a_n\}_{n=1}^{\infty}$, its Dirichlet generating function is the power series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

We also write $f(s) \overset{\text{dgf}}{\longleftrightarrow} \{a_n\}_{n=1}^{\infty}$.

Dirichlet Generating Functions

We will now briefly introduce a powerful tool used in analytic number theory.

Example

The Dirichlet generating function for the sequence $1, 1, 1, \ldots$ is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Example

The Dirichlet generating function for the sequence $\mu(1), \mu(2), \mu(3), \ldots$ is the reciprocal of the Riemann zeta functions. Where μ is the multiplicative Möbius function.

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

Dirichlet Generating Functions

Multiplicative Property

If $f(s) \overset{\text{dgf}}{\longleftrightarrow} \{a_n\}_{n=1}^{\infty}$ and $g(s) \overset{\text{dgf}}{\longleftrightarrow} \{a_n\}_{n=1}^{\infty}$, then

$$(fg)(s) \overset{\text{dgf}}{\longleftrightarrow} \left\{ \sum_{\substack{k=1 \\ k|n}}^{\infty} a_k b_{\frac{n}{k}} \right\}_{n=1}^{\infty}.$$

Questions